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LOW FREQUENCY ACOUSTIC PROPAGATION IN A WAVEGUIDE

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This report describes a method for generating contour integral representations of low frequency propagation in a range-independent waveguide. It provides several examples of evaluating the contour integral representation of the Green function for a one-and two-layered waveguide, including asymptotic evaluation of the branch cut integral.

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FOREWORD

This report presents a simple prescription for generating the appropriate contour integrals describing the Green's function for propagation in a waveguide. It then proceeds to describe methods by which these contour integrals can be evaluated in terms of sums over normal modes and branch cut integrals. Examples for the one- and two-layered waveguide are developed in detail.

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CONTENTS

INTRODUCTION

This report presents a simple prescription for generating the appropriate contour integrals describing the Green's function for propagation in a waveguide. It also describes a method by which these contour integrals can be evaluated in terms of sums over normal modes and branch cut integrals.

INTEGRAL REPRESENTATIONS AND MULTIPATHS

The first assumption is that the source is a harmonic point source with a time dependence of the form $e^{-i\omega t}$. Second, the Green function for a point source in free space is given by the expression

$$G(r,r') = e^{ik|r-r'|}/4\pi|r-r'| \tag{1}$$

which can be represented in the form

$$G(r,r') = i/8\pi\int_{-\infty}^{\infty} dh.H_0^{(1)}(q.\rho).e^{ih|z-z'|} = i/8\pi.\int_{-\infty+i0}^{+\infty-i0} \frac{qdq}{h}.H_0^{(1)}(q.\rho).e^{+ih|z-z'|} \tag{2}$$

in terms of the cylindrical functions appropriate for a waveguide. Here, q is the horizontal wavenumber and h is the vertical wavenumber of the field. The conventions of Ewing, Jardetsky and Press for the branch cut of

$$h(q) = i.\sqrt{q^2-k^2} \tag{3}$$

is chosen such that the imaginary part of h is positive on the physical sheet. For real frequencies, the branch cut consists of a cut from +k to the origin to +i∞. The cut for negative frequencies is obtained by the making the transformation $h \to -h$ along the cut. The z-axis is oriented towards the bottom of the waveguide. This convention affects the signs in the exponential of the cylindrical form of the Green function.

Using Eqn 3 for the Green function one can obtain an integral representation of the Green function in a waveguide. Consider a source and receiver in a homogenous layer, where U and V are the total reflection coefficients of the upper and lower boundaries of this layer. Then the Green function for the wave guide may be expressed in the form

$$G_w(r,r') = i/8\pi\int_{-\infty}^{+\infty} \frac{qdq}{h}.H_0^{(1)}(q.\rho).F(z,z') \tag{4}$$

where the function F(z,z') is given by the following multiple scattering summation of multipaths.

$$F(z,z') = \Sigma_{n=0}^{\infty}(e^{ih(z-z')} + U.e^{-ih(2a-z-z')} + V.e^{ih(2b-z-z')} + UV.e^{2ihd-ih.(z-z')}).(UV.e^{2ihd})^n \tag{5}$$

This summation consists of the contributions due to the direct path, a surface bounce, a bottom bounce, and a bottom bounce followed by a surface bounce and all multiple bounces. Here, a is the z coordinate of the upper surface and b is the z coordinate of the lower surface and d = b - a is the depth of the layer. In deriving the above expression it was assumed that z > z'. The expression for z < z' is obtained by switching z and z' in the above expression.

Eqn 5 can be further simplified in the form

$$F(z,z') = (e^{-ihz'} + U.e^{+ihz'})(e^{ihz} + Ve^{+2ihd}.e^{-ihz})/(1 - UVe^{2ihd}) \tag{6}$$

appropriate for a contour integral, where z > z'. The case of z < z' is represented by switching z and z' in the above expression.

The contour integral representation of the waveguide's Green function is given by the following expression.

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{+\infty}\frac{qdq}{h}.H_0^{(1)}(qp).(e^{-ihz_<} + U.e^{+ihz_<})(e^{+ihz_>} + V.e^{2ihd}.e^{-ihz_>})/(1 - U.V.e^{2ihd}) \tag{7}$$

Consider the generalization where the source and the receiver are in different layers. Let the source z' be in layer 1, where a' and b' are the z coordinates of the upper and lower surfaces. Let the receiver z be in layer 2, where a and b are the z coordinates of the upper and lower surfaces of this layer. Then using the multipath expansion of the waveguide one can express the Green function for the waveguide in the following form,

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{+\infty}\frac{qdq}{h_1}.H_0^{(1)}(q\rho).F_1(z').W_{12}.F_2(z) \tag{8}$$

where W_{12} is the total transmission coefficient between layer 1 to layer 2. Assuming z > z', the function $F_1(z')$ is obtained by summing all multipaths which strike the interface b'. The function $F_2(z)$ is obtained by summing all multipaths from the upper surface a to the receiver at z. In the case z > z', these functions are given by the following expressions.

$$F_1(z') = (e^{+ih_1(b'-z')} + U_1.e^{2ih_1d_1}.e^{-ih_1(b'-z')})/(1 - U_1.v_1.e^{2ih_1d_1}) \tag{9}$$

$$F_2(z) = (e^{+ih_2(z-a)} + V_2.e^{2ih_2d_2}.e^{-ih_2(z-a)})/(1 - U_2V_2.e^{2ih_2d_2}) \tag{10}$$

In the above expression U_1 is the total reflection coefficients at a' of all the layers above a', v_1 is the surface reflection coefficient at b' as if the layer below b' were a homogeneous halfspace, U_2 is the total reflection coefficient at a of all the layers above a, and V_2 is the total reflection coefficient at b of all the layers below b. Note the use of the vertical wavenumber in layer 1 in the function $F_1(z')$, and the use of the vertical wavenumber in layer 2 in the function $F_2(z)$.

In the case z < z', the function $F_1(z')$ is the sum of all multipaths from the source to the upper interface at a', and the function $F_2(z)$ is the sum of all multipaths from the lower interface at b to the receiver. In this case, these functions are given by the following expressions,

$$F_1(z') = (e^{-ih_1(a'-z')} + V_1.e^{2ih_1d_1}.e^{+ih_1(a'-z')})/(1 - u_1.V_1.e^{2ih_1d_1}) \tag{11}$$

$$F_2(z) = (e^{-ih_2(z-b)} + U_2.e^{2ih_2d_2}.e^{+ih_2(z-b)})/(1 - U_2V_2.e^{2ih_2d_2}) \tag{12}$$

where u_1 is the surface reflection coefficient at a' as if the layer above a' were a homogeneous halfspace, and V_1 is the total reflection coefficient at b' of all the layers below b'.

Note, this form of the integral representation is only valid in the case both the source and the field point are in fluid layers. However, the intervening layers may be elastic or Biot models, since this form of the integral representation needs only the reflection and transmission coefficient of these intervening layers. An integral representation for an elastic or Biot waveguide would require use of the propagator matrix formalism to represent the field.

NORMAL MODES

This section describes the normal mode representation one can obtain from the previous integral representations of the Green function by representing these contour integrals in terms of a sum of residues from poles and branch cut integrals. For the sake of simplicity, only the case where both the source and the receiver are in the same layer is treated.

The Green function for a source and receiver in the same layer is given by the following contour integral representations in the complex h-plane.

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{\infty} dh.H_0^{(1)}(q\rho).(e^{-ihz_<} + U.e^{+ihz_<}).(e^{+ihz_>} + V.e^{2ihd}.e^{-ihz_>})/(1 - UV.e^{2ihd}) \tag{13}$$

This integral may be represented by the contour integral in the q-plane as

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{\infty} qdq/h.H_0^{(1)}(q\rho).(e^{-ihz_<} + U.e^{+ihz_<})(e^{+ihz_>} + V.e^{+2ihd}.e^{-ihz_>})/(1 - UVe^{2ihd}) \tag{14}$$

In general the poles of the integrand lie on or near the branch cuts. In general, this contour integral may be represented as a sum of pole contributions and a cut contributions. The cut contributions occur when the integrand is not an even function of h. In the case U = -1 and V = +1 for a pressure release surface and a rigid bottom, the integrand is an even function of h, hence the cut contribution is zero. In the case U and V have the symmetry $U(-h) = U(h)^{-1}$ and $V(-h) = V(h)^{-1}$, the integrand is an even function of h and its branch cut contribution vanishes. In general, the cut contribution due to any bounded layer is zero. Cut contributions generally occur when you have an unbounded region at the ends of the waveguide.

The branch cut for $q = \sqrt{k^2 - h^2}$ in the complex h-plane is chosen such that Im(q) > 0 in the complex h-plane. The above contour can be completed by adding the contour at infinity in the upper half plane, since the integrand is exponential damped along the added contour. In addition to the branch cut for q, there exists branch cuts for the vertical wavenumber in the other layers, and if there exists an unbounded region in the waveguide there will be a cut contribution to the above integrand in addition to the pole contributions.

The poles of the above integrand correspond to the complex solutions of the equation.

$$(1 - UV.e^{2ihd}) = 0 \tag{15}$$

The pole contributions to the above integrand are of the form

$$\tfrac{1}{4}.H_0^{(1)}(q\rho).(e^{-ihz'} + U.e^{+ihz'}).(e^{-ihz} + U.e^{+ihz})/(U.\Delta') \tag{16}$$

where

$$\Delta' = \tfrac{\partial}{\partial h}(1 - UVe^{2ihd}) = -2i.(d + \tfrac{1}{2i}.\tfrac{1}{UV}.\tfrac{\partial}{\partial h}(UV)) \tag{17}$$

The normal mode representation of the Green function is simply the sum over all the residues from the poles and cut contributions in the upper half q-plane.

HOMOGENEOUS WAVEGUIDE WITH A RIGID BOTTOM

This section derives the Green's function for a homogeneous waveguide with a pressure release surface and a rigid bottom. The reflection coefficients U and V are given by the expressions U = -1 and V = +1. Substituting these values into Eq. 16 for the pole contributions to the Green function the following expression is obtained,

$$-i/2d.H_0^{(1)}(q\rho).(e^{+ihz'} - e^{-ihz'})(e^{+ihz} - e^{-ihz})$$

which may be represented in the form

$$i/4.H_0^{(1)}(q\rho)f(z').f(z) \tag{18}$$

where

$$f(z) = \sqrt{\tfrac{2}{d}}\ \sin(hz) \tag{19}$$

are the normalized depth function of the waveguide.

The characteristic equation for the waveguide takes the form

$$(1 + e^{2ihd}) = 0 \tag{20}$$

whose solutions are given by the expression

$$h = (n - 1/2) * \pi/d \tag{21}$$

for n = 1,2...

TWO LAYERED WAVEGUIDE WITH RIGID BOTTOM

This section applies the previous sections in the case the waveguide consists of 2 fluid layers, where the top layer has a pressure release surface and the bottom layer has a rigid bottom. In this case the reflection coefficients at the surface and bottom are given by the expressions.

$$U_1 = u_1 = -1 \tag{22}$$

$$V_2 = v_2 = +1 \tag{23}$$

The reflection coefficients at the interface of these 2 layers is more complicated. Note that the surface reflection coefficients at this interface are given by the expressions,

$$v_1 = (m.h_1 - h_2)/(m.h_1 + h_2) \tag{24}$$

$$u_2 = -v_1 \tag{25}$$

where m is the ratio of the density in layer 2 and the density in layer 1. The total reflection coefficients at this interface are given by the expressions

$$V_1 = (v_1 + V_2.e^{+2ih_2d_2})/(1 + v_1 V_2.e^{+2ih_2d_2}) = (v_1 + e^{+2ih_2d_2})/(1 + v_1.e^{+2ih_2d_2}) \tag{26}$$

$$U_2 = (u_2 + U_1.e^{+2ih_1d_1})/(1 + u_2 U_1.e^{+2ih_1d_1}) = (-v_1 - e^{2ih_1d_1})/(1 + v_1 e^{+2i_1hd_1}) \tag{27}$$

In the case both the source and receiver are in the upper layer (layer 1), the Green function is given by the following contour integral.

$$G_w(r, r') = i/8\pi \int_{-\infty}^{+\infty} \frac{qdq}{h_1} (e^{-ih_1z'} - e^{+ih_1z'})(e^{+ih_1z} + V_1.e^{+2ih_1d_1}.e^{-ih_1z})/(1 + V_1.e^{+2ih_1d_1}) \tag{28}$$

The normal mode (pole) contribution to this integral is given by the expression,

$$-1/4.\frac{1}{\Delta'}.H_0^{(1)}(q\rho).(e^{+ih_1z'} - e^{-ih_1z'})(e^{+ih_1z} - e^{-ih_1z}) \tag{29}$$

where

$$\Delta' = -2i(d_1 + \frac{1}{V_1}.\frac{\partial}{\partial h_1}(V_1)) \tag{30}$$

and the normal modes (poles) are solutions of the following equation.

$$\Delta = (1 + V_1 . e^{+2ih_1 d_1}) = 0 \tag{31}$$

Multiplying the characteristic equation by $(mh_1 + h_2)(1 + v_1 . e^{+2ih_2 d_2}).e^{-ih_1 d_1}.e^{-ih_2 d_2}$ one can express the poles (normal modes) as solutions of the equation.

$$\Upsilon = 2.(mh_1 - h_2)\cos(h_1 d_1 - h_2 d_2) + 2.(mh_1 + h_2) * \cos(h_1 d_1 + h_2 d_2) \tag{32}$$

Note, this function is a real odd function of the vertical wavenumber in both the first and second layer, which implies that if $h_1 = \alpha$ is a solution of this equation, then $-\alpha, \alpha*, -\alpha*$ are also solutions of the characteristic equation. The roots to this equation are real for real wavenumbers k_1, k_2. Hence the roots lie along the branch cuts for the vertical wavenumber.

In order to compute the derivative of the characteristic equation define the quantity

$$\gamma^2 = k_1^2 - k_2^2 \tag{33}$$

The vertical wavenumber in the second layer maybe rewritten in the form

$$h_2 = \sqrt{h_1^2 - \gamma^2} \tag{34}$$

in terms of the wavenumber in the first layer. The derivative of the characteristic equation evaluated at a solution of the characteristic equation is given by the expression.

$$\Delta' = -2id_1 - \frac{1}{h_2}\{2id_2.h_1.((mh_1 + h_2) + (mh_1 - h_2).e^{+2ih_1 d_1}) + ((mh_2 + h_1) + (mh_2 - h_1).e^{+2ih_2 d_2})$$

$$+((mh_2 - h_1) + (mh_2 + h_1).e^{+2ih_2 d_2}).e^{+2ih_1 d_1}\}/((mh_1 + h_2) + (mh_1 - h_2).e^{+2ih_2 d_2}) \tag{35}$$

In the homogeneous limit, $m = 1$, $h_2 = h_1 = h$, the characteristic equation and its derivative are given by the expressions

$$\Delta = (1 + e^{+2ih(d_1 + d_2)}) = 0 \tag{36}$$

and

$$\Delta' = -2i.(d_1 + d_2) \tag{37}$$

appropriate for a homogeneous waveguide with a rigid bottom.

In the case the source z' is in the top layer, and the field point z is in the bottom layer, the Green function is given by the following expression,

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{+\infty} \frac{qdq}{h_1} H_0^{(1)}(q\rho).F_1(z').W_{12}.F_2(z) \tag{38}$$

where

$$W_{12} = (1+v_1) \tag{39}$$

is the transmission coefficient between layers 1 and 2, and

$$F_1(z') = (e^{+ih_1(z_2-z')} - e^{+2ih_1d_1}.e^{-ih_1(z_2-z')})/(1+v_1.e^{+2ih_1.d_1}) \tag{40}$$

and

$$F_2(z) = (e^{+ih_2(z-z_2)} + e^{+2ih_2d_2}.e^{-ih_2(z-z_2)})/(1-U_2.e^{+2ih_2d_2}). \tag{41}$$

Here, $z_1 = 0, z_2 = +d_1, z_3 = (d_1+d_2)$ are the z coordinates of the three interfaces of the waveguide. By making use of the identity,

$$(1+v_1.e^{+2ih_1d_1}).(1-U_2.e^{+2.i.h_2d_2}) = (1+v_1.e^{+2ih_2d_2}).(1+V_1.e^{+2ih_1d_1}) \tag{42}$$

the expression for the Green function can be expressed
in the following form.

$$G_w(r,r') = -1/2\pi.\int_{-\infty}^{\infty}.\frac{qdq}{h_1}.H_0^{(1)}(q\rho).\{(1+v_1)/(1+v_1.e^{+2ih_2d_2})\}.e^{+ih_1d_1+ih_2d_2}/\Delta$$

$$\sin(h_1z').\cos(h_2(z-d_1-d_2)) \tag{43}$$

The normal modes are solutions of the characteristic equation

$$\Delta = (1+V_1.e^{+2ih_1d_1}) = 0 \tag{44}$$

and the pole contribution to this integral is of the form.

$$(+i/\Delta').(\{(1+v_1)/(1+v_1.e^{+2ih_2d_2})\}.e^{+ih_1d_1+ih_2d_2}.H_0^{(1)}(q\rho).\sin(h_1z').\cos(h_2(z-d_1-d_2)) \tag{45}$$

This solution can be recast into the form

$$i/4.H_0^{(1)}(q\rho).f_1(z').f_2(z) \tag{46}$$

where $f_1(z)$ and $f_2(z)$ are the functions

$$f_1(z) = \sqrt{-4i/\Delta'} \,.\, \sin(h_1 z) \tag{47}$$

and

$$f_2(z) = \sqrt{\tfrac{-4i}{\Delta'}} \,.(+i)\{(1+v_1)/(1+v_1.e^{+2ih_2d_2})\}.e^{+ih_1d_1+ih_2d_2}.\cos(h_2.(z-d_1-d_2)). \tag{48}$$

Similarly, the solution in the case both the source and field point are contained within layer 1 can be recast into the form.

$$i/4.H_0^{(1)}(q\rho).f_1(z').f_1(z) \tag{49}$$

In the homogeneous limit, $m = 1$, $h_1 = h_2 = h$, the depth functions are equal to the following expression,

$$f_1(z) = f_2(z) = \sqrt{\tfrac{2}{(d_1+d_2)}} \,.\, \sin(hz) \tag{50}$$

appropriate for a homogeneous waveguide.

FLUID LAYER OVER A HOMOGENEOUS HALFSPACE

This section will describe the Green's function for a fluid layer over a fluid halfspace. In this case, the reflection coefficients are

$$U_1 = u_1 = -1 \tag{51}$$

$$V_1 = v_1 = (mh_1 - h_2)/(mh_1 + h_2) \tag{52}$$

$$u_2 = -v_1 \tag{53}$$

$$U_2 = -(v_1 + e^{+2ih_1d_1})/(1 + v_1.e^{+2ih_1d_1}) \tag{54}$$

$$V_2 = v_2 = 0 \tag{55}$$

where the conventions of the previous section have been adopted.

In the case z > z', and the source and the receiver are contained within layer 1, the Green's function is given by the contour integral.

$$G_w(r,r') = i/8\pi.\int_{-\infty}^{+\infty}.\tfrac{qdq}{h_1}.H_0^{(1)}(q\rho).(e^{-ih_1z'} - e^{+ih_1z'})(e^{+ih_1z} + v_1.e^{+2ih_1d_1}.e^{-ih_1z})/(1+v_1e^{+2ih_1d_1}) \tag{56}$$

The normal modes are solutions of the characteristic equation,

$$\Delta = (1 + \nu_1 . e^{+2ih_1 d_1}) = 0 \qquad (57)$$

which can be recast into the form.

$$\Delta = ((mh_1 + h_2) + (mh_1 - h_2) . e^{+2ih_1 d_1}) / (mh_1 + h_2) = 0 \qquad (58)$$

The normal mode contribution to this contour integral is given by the residue

$$\frac{1}{\Delta'} . H_0^{(1)}(q\rho) . \sin(h_1 z') . \sin(h_1 z) , \qquad (59)$$

where

$$\Delta' = -2id_1 - \frac{1}{\nu_1}\frac{\partial}{\partial h_1}(\nu_1) = -2id_1 - \frac{1}{h_2} . \{(mh_2 - h_1)/(mh_1 + h_2) - (mh_2 + h_1)/(mh_1 + h_2)\} \qquad (60)$$

is the derivative of the characteristic equation evaluated at the pole.

The integrand is an even function of h_1, but it is not an even function of h_2 This leads to a discontinuity across the h_2 cut in the complex q-plane, which leads to a branch cut contribution to the above contour integral, in addition to the pole contributions. This branch cut contribution is due to the evanescent fields generated by the halfspace, and it is of the form,

$$\int_0^{+\infty} h_2 . dh_2 . (F(+h_1, +h_2) - F(-h_1, -h_2)) \qquad (61)$$

along the branch cut of h_2 and $F(h_1, h_2)$ is the function.

$$F(h_1, h_2) = (i/8\pi)\frac{1}{h_1} H_0^{(1)}(q\rho) . (e^{-ih_1 z'} - e^{+ih_1 z'})(e^{+ih_1 z} + \nu_1 . e^{+2ih_1 d_1} . e^{-ih_1 z}) / (1 + \nu_1 . e^{+2ih_1 d_1}) \qquad (62)$$

Making use of the identities,

$$\nu_1(-h_2) = +1/\nu_1(+h_2) \qquad (63)$$

$$\nu_1(-h_1) = +1/\nu_1(+h_1) \qquad (64)$$

this contour integral can be written in the form.

$$-i/2\pi . \int_0^{+\infty} dh_2 . h_2 / h_1 . H_0^{(1)}(q\rho) . \sin(h_1 z') . \sin(h_1 z) . (\nu_1^2 - 1) / ((\nu_1 + e^{-2ih_1 d_1})(\nu_1 + e^{+2ih_1 d_1}))$$

$$= +i/2\pi . \int_0^{+\infty} dh_2 . H_0^{(1)}(q\rho) . \sin(h_1 z') . \sin(h_1 z) . m . h_2^2 / ((mh_1 \cos(h_1 d_1))^2 + (h_2 . \sin(h_1 d_1))^2) \qquad (65)$$

The integrand is an even function of h_1 and h_2.

Assuming $k_2\rho >> 1$ and using the relationship

$$q = +i\sqrt{h_2^2 - k_2^2} \tag{66}$$

the integrand is exponentially damped for h_2 greater than k_2 due to the exponential behavior of the Hankel function for large arguments. In this case, the chief contribution to the integrand comes from the oscillatory portion of the integrand from $h_2 = 0$ to k_2.

Assuming $k_2\rho >> 1$, the branch line integral can be re-expressed in the form,

$$\int_0^{+\infty} h_2^2.dh_2.e^{+iq\rho}.F(h_2) \rightarrow \tfrac{1}{2}.\Gamma(3/2).(2k_2/\rho)^{3/2}.e^{+ik_2\rho - i.3\pi/4}.F(h_2 = 0) \tag{67}$$

where the asymptotic expansion (Eq. 68) has been used

$$H_0^{(1)}(q\rho) \rightarrow \sqrt{\tfrac{2}{\pi.q.\rho}}\,.e^{+iq\rho - i\pi/4} \tag{68}$$

for the Hankel functions, and the method of steepest descent to evaluate integrals of the form

$$\int_{-\infty}^{+\infty} x^n.dx.e^{+if(x)}.F(x) \rightarrow \Gamma((n+1)/2).|f''(x_0)/2|^{-(n+1)/2}.e^{+i.f(x_0) \pm i(n+1).\pi/4}.F(x_0) \tag{69}$$

where x_0 is the saddle point satisfying the equation, $f'(x_0) = 0$. The asymptotic expression for the branch line integral is given by the expression,

$$-i.k_2/(2\pi m).e^{+ik_2\rho}/(\gamma\rho)^2.\sin(\gamma z').\sin(\gamma z)/\cos(\gamma d_1)^2 \tag{70}$$

where $\gamma = \sqrt{k_1^2 - k_2^2}$. Note in particular, that the asymptotic form of the branch line integral falls off as ρ^{-2}, whereas the asymptotic form of the normal mode contributions fall off as $\rho^{-1/2}$. Thus, the branch line integral is negligible at far ranges, where the normal mode terms dominate.

Let us next consider the case the source is in layer 1 and the field point is in layer 2. In this case, the Green's function is given by the expression

$$i/8\pi.\int_{-\infty}^{+\infty} \tfrac{q\,dq}{h_1}.H_0^{(1)}(q\rho).(e^{-ih_1z'} - e^{+ih_1z'}).e^{+ih_2(z-d_1)}.(1+\nu_1).e^{+ih_1d_1}/(1+\nu_1.e^{+2ih_1d_1}). \tag{71}$$

The normal mode contribution to this integral is of the form

$$-i/2\pi.(1+\nu_1).e^{+ih_1d_1}/\Delta'.H_0^{(1)}(q\rho).\sin(h_1(z').e^{+ih_2(z-d_1)}. \tag{72}$$

The branch cut integral is of the form

$$i/2\pi.\int_0^{+\infty} h_2 dh_2/h_1.H_0^{(1)}(q\rho).\sin(h_1 z')(1+v_1).\{v_1.\sin(h_2(z-d_1)-h_1 d_1)+\sin(h_2(z-d_1)+h_1 d_1)\}$$

$$.e^{+2ih_1 d_1}/\{(1+v_1.e^{+2ih_1 d_1}).(v_1+e^{+2ih_1 d_1})\}. \tag{73}$$

This can be rewritten in the form

$$i/2\pi.\int_0^{+\infty} h_2.dh_2.H_0^{(1)}(q\rho).\sin(h_1 z').\{mh_1.\sin(h_2(z-d_1)).\cos(h_1 d_1)+h_2\cos(h_2(z-d_1))\sin(h_1 d_1)\}$$

$$.m/\{(mh_1\cos(h_1 d_1))^2+(h_2\sin(h_1 d_1))^2)\}. \tag{74}$$

This integral is of the form

$$\int_0^{+\infty} h_2^2.dh_2.e^{+iq\rho}.F(h_2) \tag{75}$$

where

$$F(h_2)=i/2\pi.\sqrt{\frac{2}{\pi.q.\rho}}.e^{-i\pi/4}.m.\sin(h_1 z').\{mh_1/h_2.\sin(h_2(z-d_1)).\cos(h_1 d_1)+$$

$$\cos(h_2.(z-d_1)).\sin(h_1 d_1)\}/\{(mh_1\cos(h_1 d_1))^2+(h_2\sin(h_1 d_1))^2)\} \tag{76}$$

An asymptotic expression for this branch cut integral may be obtained as in the previous case. The resulting asymptotic expression is of the form

$$-i.k_2/(2\pi.m).e^{+ik_2\rho}/(\gamma\rho)^2.\sin(\gamma z')\{\sin(\gamma d_1)+m\gamma.(z-d_1).\cos(\gamma d_1)\}/\cos(\gamma d_1)^2. \tag{77}$$

Note, this term changes linearly with respect to the depth of the field point.

REFERENCES

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